An implementation of the X-FEM for Eulerian solid-mechanics

MULTIMAT

Arcachon, France

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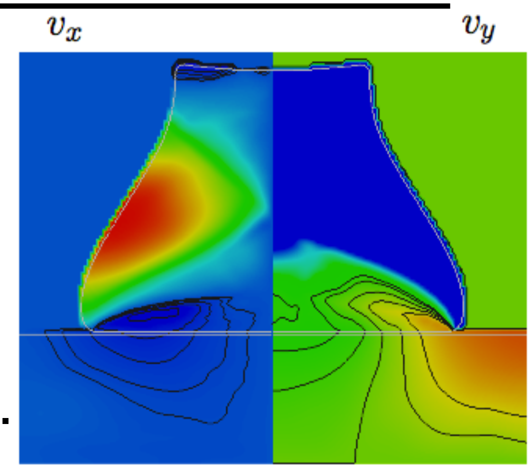
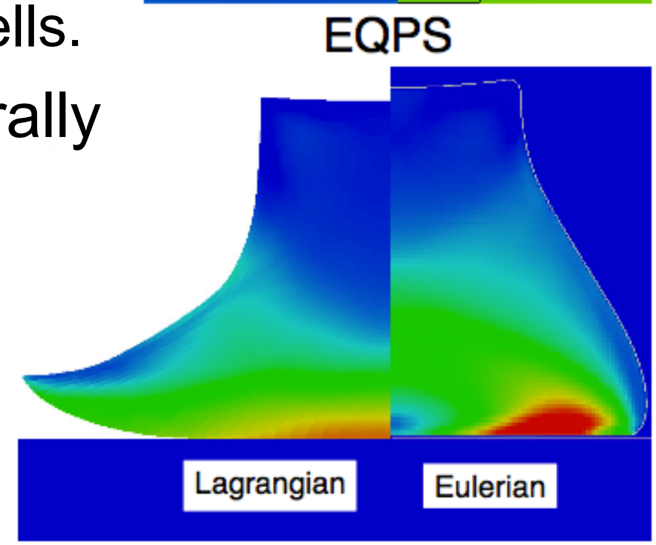
We use Eulerian approach but mixed material cells can be problematic:

- Multi-material problems with significant vorticity/distortion:
  - Lagrangian approaches tangle.
  - Arbitrary Lagrangian Eulerian (ALE) can't merge materials without topology changes.
  - Eulerian approach produces mixed-cells.
- Current mixed-cell approaches generally assume materials are "well" mixed:
  - Assume "equilibrated" state
  - Single velocity/displacement field
  - Lack of intra-element interfaces





Lagrangian step requires closure model(s) for mixed-cell properties:

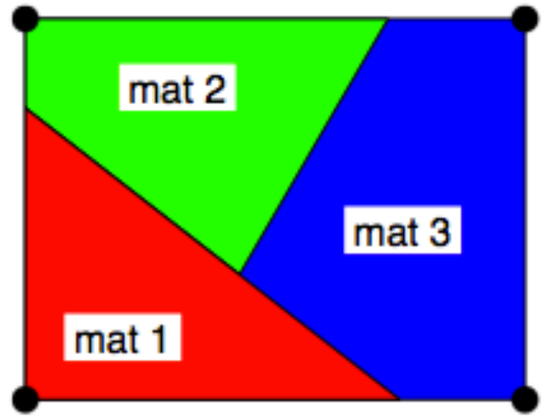
$$\mathbf{a}^n = \mathbf{M}^{-1}\left[\mathbf{f}^n_{hg} + \mathbf{f}^n_{ext} - \int_\Omega \mathbf{B}^t \sigma^n \right]$$

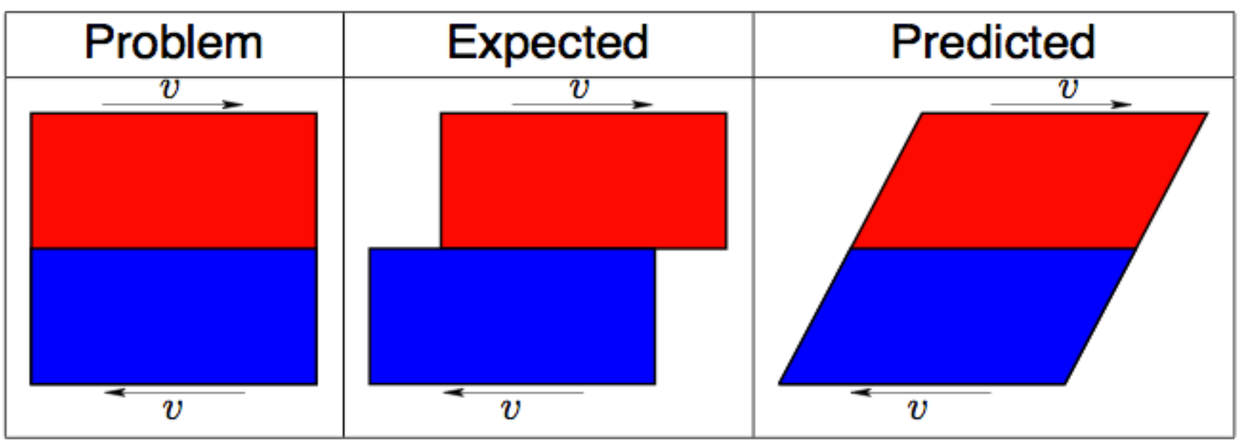
$$\mathbf{v}^{n+1/2} = \mathbf{v}^{n-1/2} + \overline{\Delta t}\,\mathbf{a}^n \quad \mathbf{x}^{n+1} = \mathbf{x}^n + \overline{\Delta t}\,\mathbf{v}^{n+1/2}$$

$$\mathbf{D}^{n+1/2} = \frac{1}{2}\left(\mathbf{L}^t + \mathbf{L}\right)^{n+1/2} \quad \mathbf{D}_m = \mathcal{F}\left(\mathbf{D}, etc...\right)$$

$$\sigma^{n+1}_m = \mathcal{M}_m\left(\sigma^n_m, \mathbf{D}^{n+1/2}_m, etc...\right) \qquad e^{n+1} = e^n + \overline{\Delta t} m^{-1} \int_\Omega \sigma^n_m : \mathbf{D}^{n+1/2}_m$$

$$\sigma = \mathcal{G}\left(\sigma_m, etc...\right)$$





Why X-FEM:

- Mechanism for intra-element material interfaces.
- Retains base FEM convergence properties.
- Large literature base for X-FEM in context of large deformation, explicit Lagrangian mechanics.
- Beginning to be adapted to “operator-split” multi-material Eulerian solid-mechanics [VB06; DLZRM10]:
  - explicit (central difference) Lagrangian solve,
  - followed by data transfer “remap” to “better” mesh.
- Can be incorporated into existing explicit central-difference strength/hydrodynamics codes (ALEGRA).

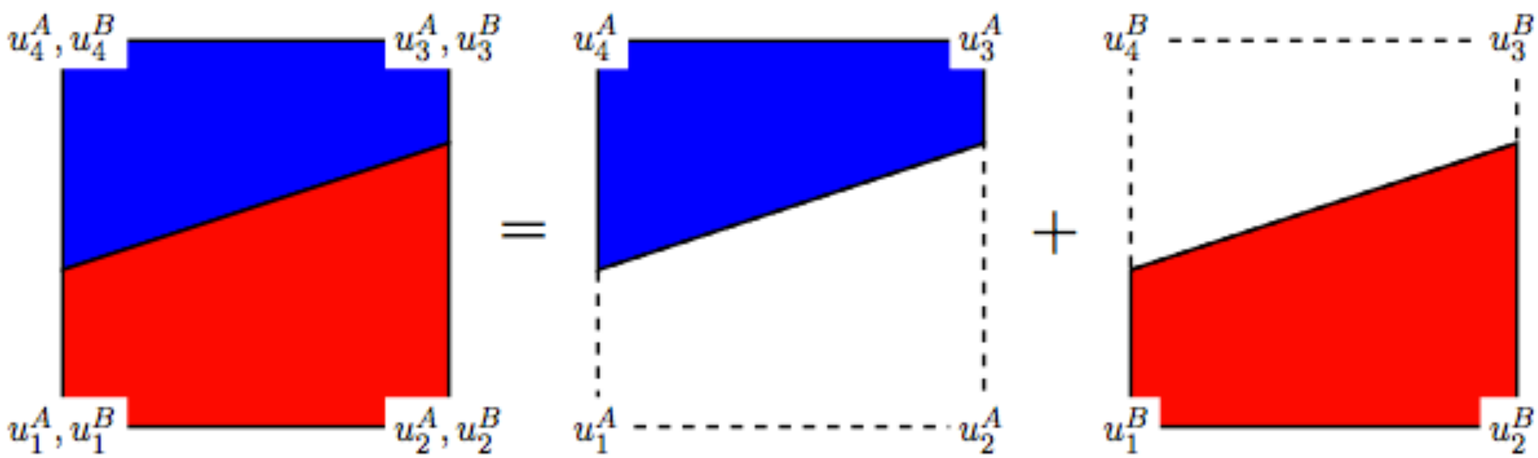
We follow the XFEM decomposition approach [HH04, SAB06] …

Multi-material enriched element equivalent to multiple single-material elements:

$$u^h(\mathbf{x}) = \cancel{\sum_I u_I^0(\mathbf{x}) N_I(\mathbf{x}) +} \sum_m \sum_J u_J^m N_J(\mathbf{x}) H_m(\mathbf{x})$$

$$u^h(\mathbf{x}) = u_A^h(\mathbf{x}) + u_B^h(\mathbf{x})$$

$$u_m^h(\mathbf{x}) = \sum_J u_J^m N_J(\mathbf{x}) H_m(\mathbf{x})$$



Explicit central difference discretization requires care for stability …

$$\mathbf{M}\mathbf{a}^n = \mathbf{f}_{int}^n + \mathbf{f}_c$$

- lumped mass matrix with uniform partitioning of element mass to nodes [MRMCB08]

$$\mathbf{M}_m^e = \left(\rho_m^e A^e \phi_m^e / 4\right)\mathbf{I}_{8\times 8} \qquad \phi_m^e = A_m^e \big/ A^e$$

- matched with gradient operator mean quadrature [SAB06] $\bar{\mathbf{B}} = \int_{\Omega_e} \mathbf{B}\, d\Omega \Big/ A_e \qquad f_{int,m} = \bar{\mathbf{B}}^t \bar{\sigma}_m A^e \phi_m^e$

- and constraint enforcement between X-FEM interfaces:

$$\mathbf{f}_c = ?$$

We understand the issues but use "node-segment-like" Lagrange Multipliers…

… in an attempt to:

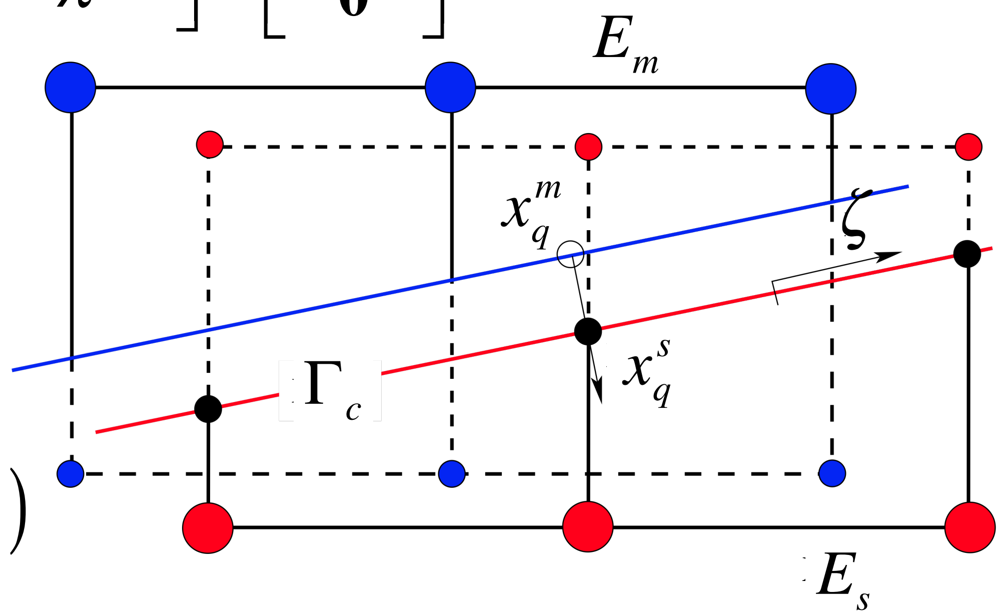
- minimize interpenetration of X-FEM interfaces,
- and retain a finite stable time-step.

- Other options for explicit X-FEM:
  - Merge (small time step) [VB06]
  - Mortar lagrange multiplier (need care for LBB)
  - Penalty (overlap, mass modifications) [DLZRM10]
  - Nitsche's (overlap, mass modifications) [AHD11]
  - Vital Vertices LM [BMW09, HAD11]

… so we use it anyway for it's practicality and economy.

Forward Increment Lagrange Multiplier approach [CTK91] …

$$\begin{bmatrix} \mathbf{M} & \Delta t\mathbf{G}^t \\ \mathbf{G} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{v}^{n+1/2} \\ \lambda \end{bmatrix}=\begin{bmatrix} \mathbf{f}^n \\ \mathbf{0} \end{bmatrix}$$



Algorithm:

$$\mathbf{v}_0^{n+1/2}=\mathbf{M}^{-1}\mathbf{f}^n$$

$$\mathbf{r}_i=\mathbf{G}\mathbf{v}_i^{n+1/2}$$

$$\lambda_{i+1}=\lambda_i+\mathbf{H}\mathbf{r}_i \quad \mathbf{H}\approx\left(\mathbf{G}\mathbf{M}^{-1}\mathbf{G}^t\right)$$

$$\mathbf{v}_{i+1}^{n+1/2}=\mathbf{M}^{-1}\left(\mathbf{f}^n-\Delta t\mathbf{G}^t\lambda_{i+1}\right)$$

No additional limitations to stable time step [DLZRM10,VMR10]

Data transfer can be accomplished in a number of ways:

- With the goal of preserving some key features:
  - Conserve mass, momentum and internal energy.
  - Do not create new minima and maxima (monotonicity).
  - Volume fractions sum to one after remap.
- Options include:
  - Interpolation (violates conservation) [DLZRM10]
  - Projection methods (violates conservation and monotonicity)
  - Geometric intersection with Van Leer limiting
    - conservation is built in.
    - limiting preserves monotonicity.

Geometric intersection with Van Leer limiting in two dimensions (1):

- Taylor Series provides functional form on donor mesh:

$$f(\mathbf{x}) = \boxed{\boxed{\boxed{\bar{f}_e} + (\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{G}_e} + \frac{1}{2}(\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{H}_e(\mathbf{x}-\mathbf{x}_e)^t - \chi}$$

first-

second-

third-order

χ provides conservation:

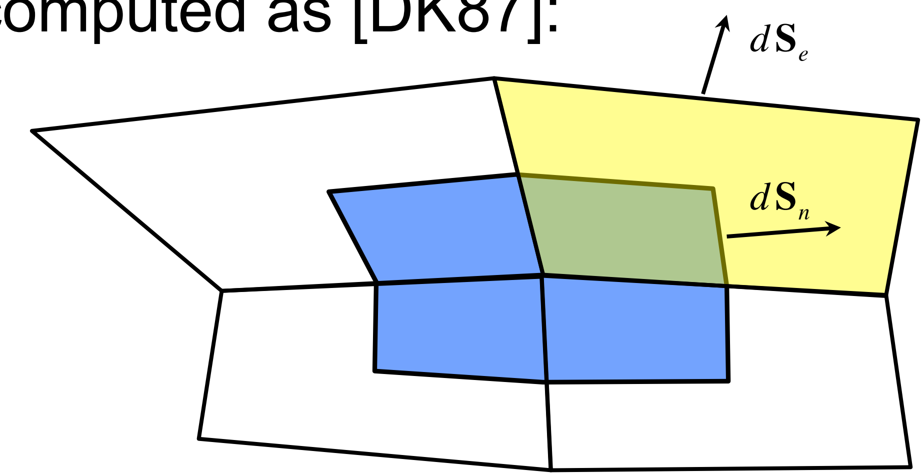
$$\bar{f}_e A_e = \int f(\mathbf{x})\,d\Omega_e \Rightarrow \chi = \frac{1}{2A_{e\cap m}}\int(\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{H}_e(\mathbf{x}-\mathbf{x}_e^t)\,d\Omega_e$$

- Gradients/hessians computed as [DK87]:

$$\mathbf{G}_n = \frac{1}{A_n}\oint \bar{f}_e\, d\mathbf{S}_n$$

$$\mathbf{G}_e = \frac{1}{A_e}\sum_n A_{n\cap e}\mathbf{G}_n$$

$$\mathbf{H}_e = \frac{1}{A_e}\oint \mathbf{G}_n\, d\mathbf{S}_e$$



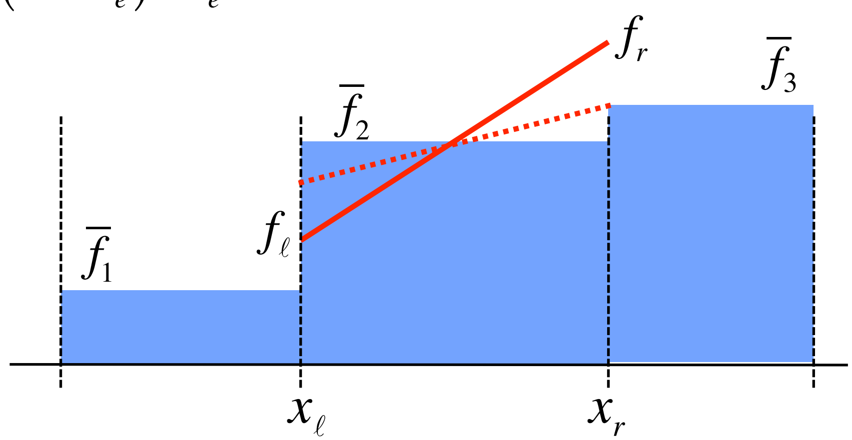
Geometric intersection with Van Leer limiting in two dimensions (2):

- Scale gradient to enforce monotonicity:

$$f(\mathbf{x}) = \bar{f}_e + s_G(\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{G}_e$$

$$s_{G,r} = \begin{cases} 1 & \text{if } \left(f_r - \bar{f}_2\right)\left(\bar{f}_3 - f_r\right) \geq 0 \\ \left(\bar{f}_3 - \bar{f}_2\right)/\left(f_r - \bar{f}_2\right) & \text{otherwise} \end{cases}$$

$$s_{G,\ell} = \begin{cases} 1 & \text{if } \left(f_\ell - \bar{f}_2\right)\left(\bar{f}_1 - f_\ell\right) \geq 0 \\ \left(f_1 - f_2\right)/\left(f_\ell - f_2\right) & \text{otherwise} \end{cases}$$



$$s_G = \max\left(\min(s_{G,\ell}, s_{G,r}), 0\right)$$

- If third-order also scale hessian terms:

$$f(\mathbf{x}) = \bar{f}_e + s_G(\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{G}_e + s_H\left[(\mathbf{x}-\mathbf{x}_e)^t\,\mathbf{H}_e(\mathbf{x}-\mathbf{x}_e)^t - \chi\right]$$

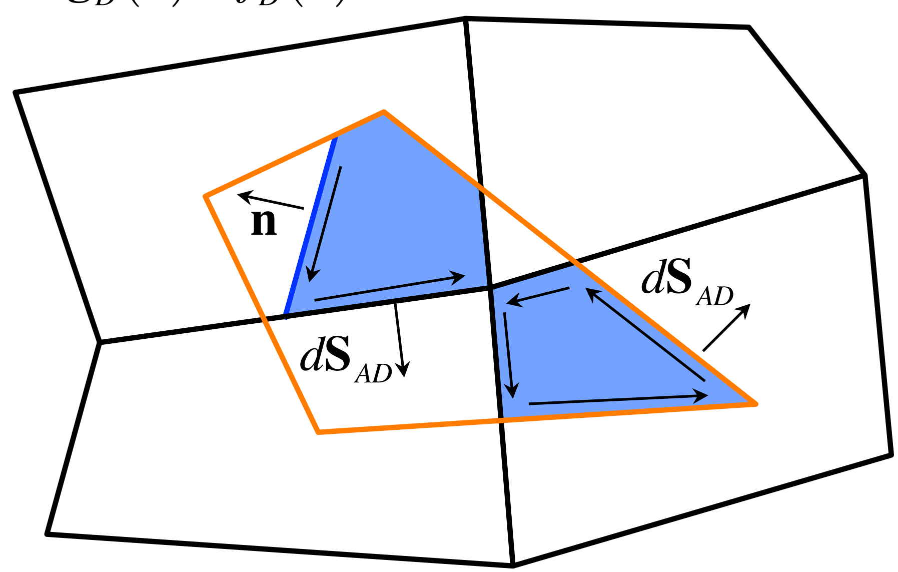
Geometric intersection with Van Leer limiting in two dimensions (3):

- Integrate function over donor-acceptor intersection element and accumulate to acceptor [D83].

$$\bar{f}_A = \frac{1}{A_A} \sum_D \int_{A_D \cap A_A} f_D(\mathbf{x})\, dA = \frac{1}{A_A} \sum_D \oint_{\Gamma_{AD}} \mathbf{g}_D(\mathbf{x})\, d\mathbf{S}_{AD}$$

$$\nabla^t \mathbf{g}_D(\mathbf{x}) \equiv f_D(\mathbf{x})$$

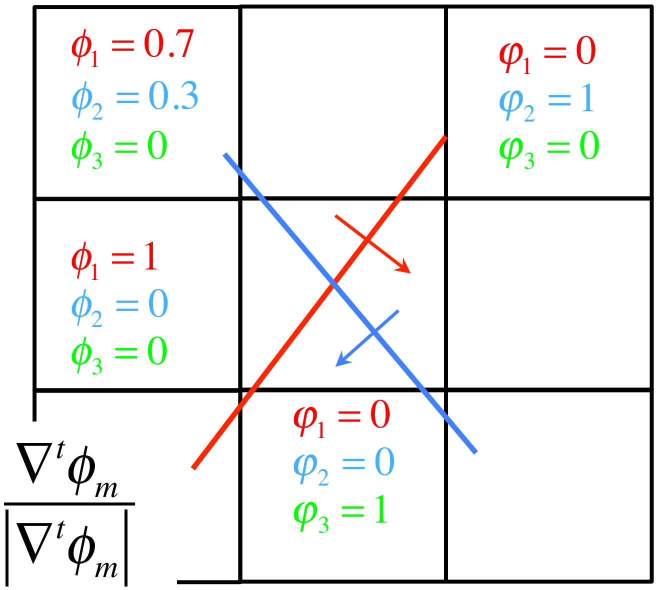
- Further restrict integral to filled region of donor mesh.

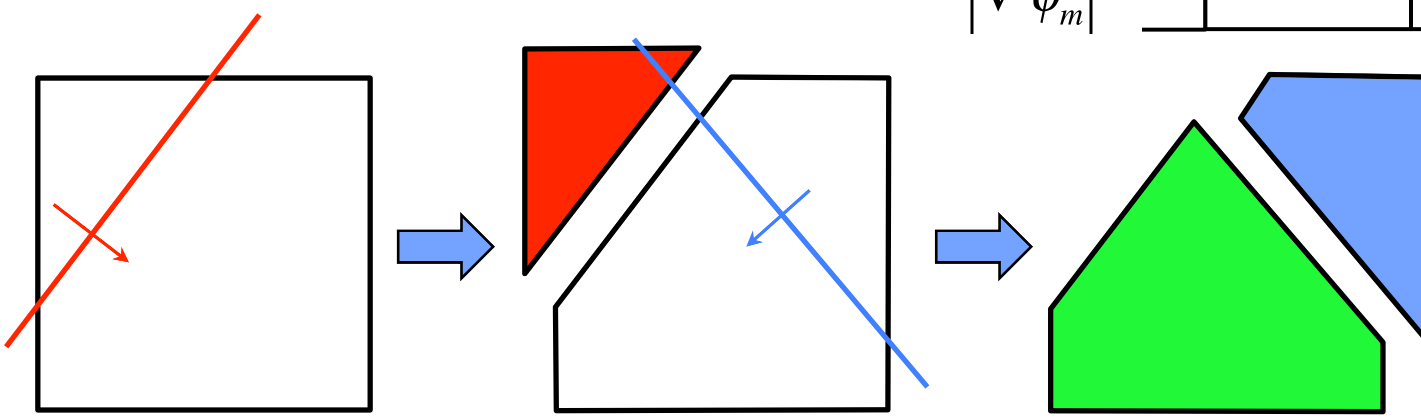


Use interface reconstruction rather than level-set approach: [DVMR08]

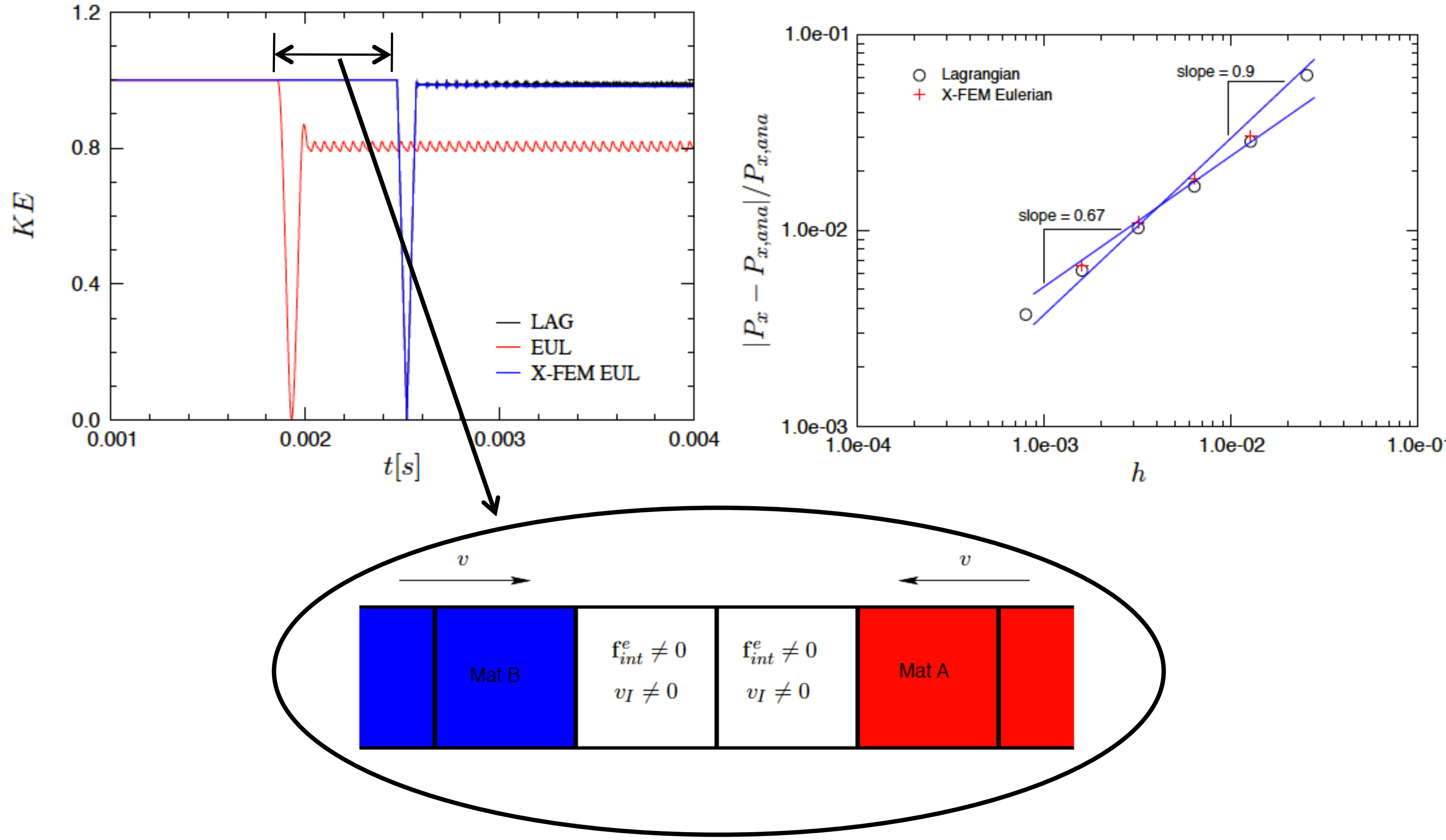
- Interfaces rebuilt after remap step.
- Using VOF approach:
  - Compute material volume-fraction gradients.
  - Reposition interface along normal to match volume.
  - Remove material from cell.

$$\mathbf{n}_m = \frac{\nabla^t \phi_m}{\left|\nabla^t \phi_m\right|}$$



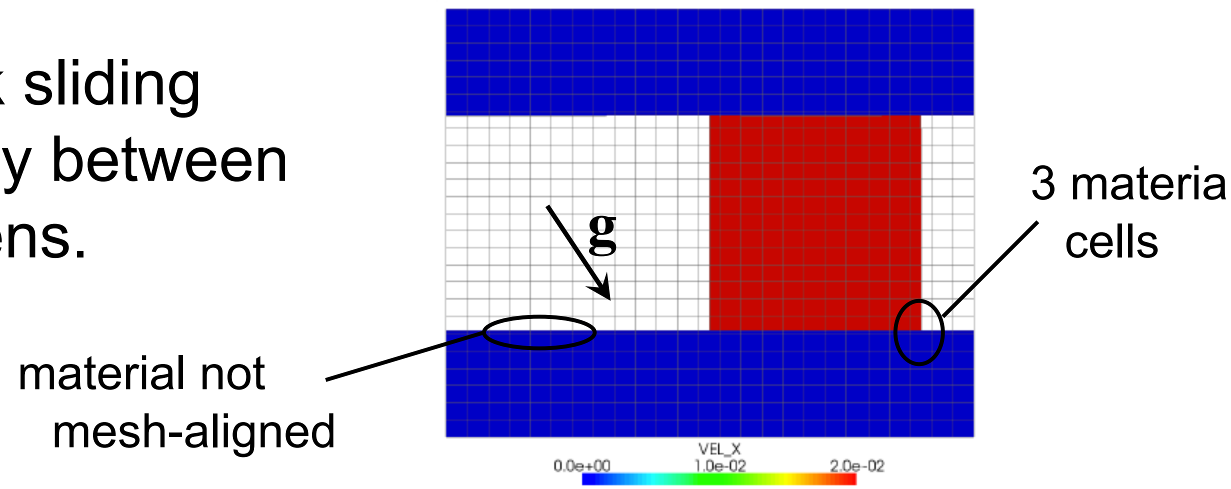


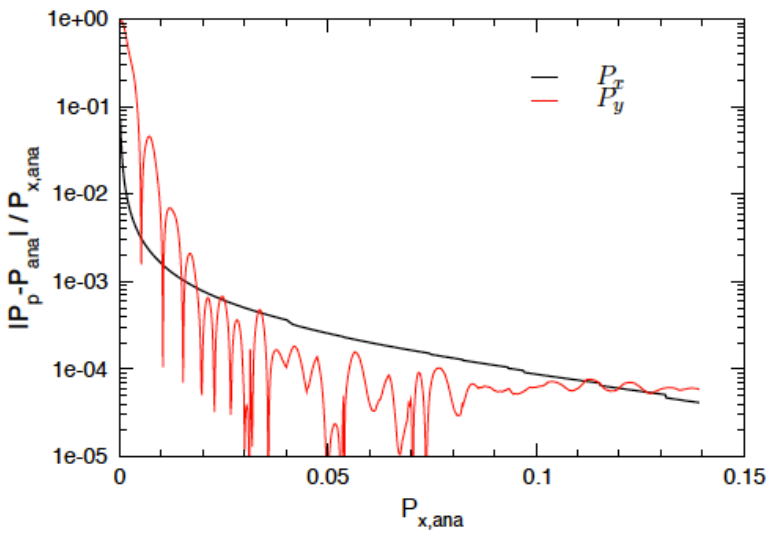
Provides enhanced results for simple one-dimensional problem [CTK91]…



… and simple two-dimensional problems …

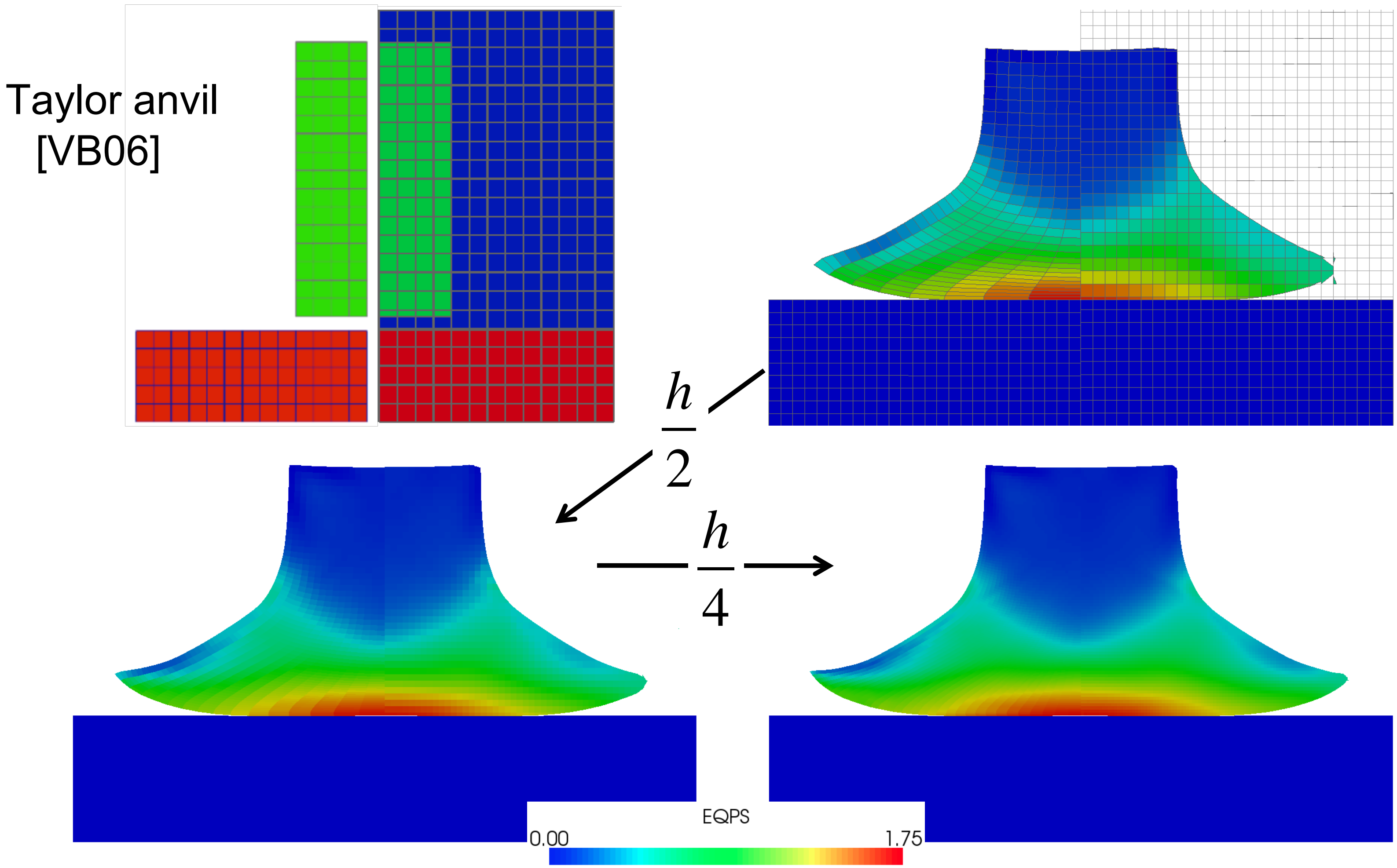
“Rigid” block sliding frictionlessly between “rigid” platens.



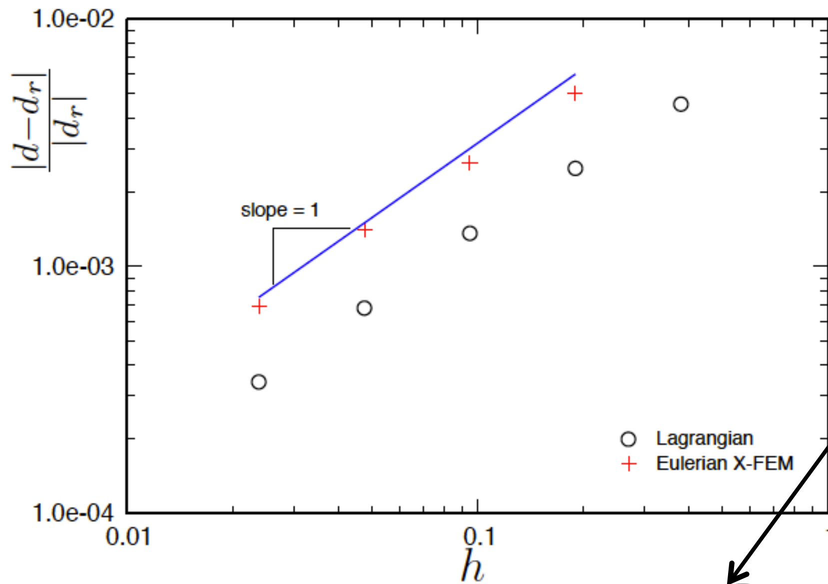


Excellent agreement for momentum compared to analytical solution.

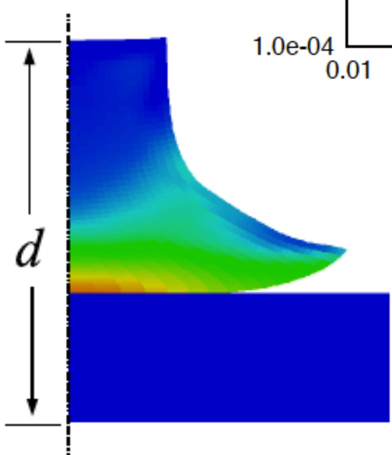
… and accuracy comparable to Lagrangian for more complex problems …

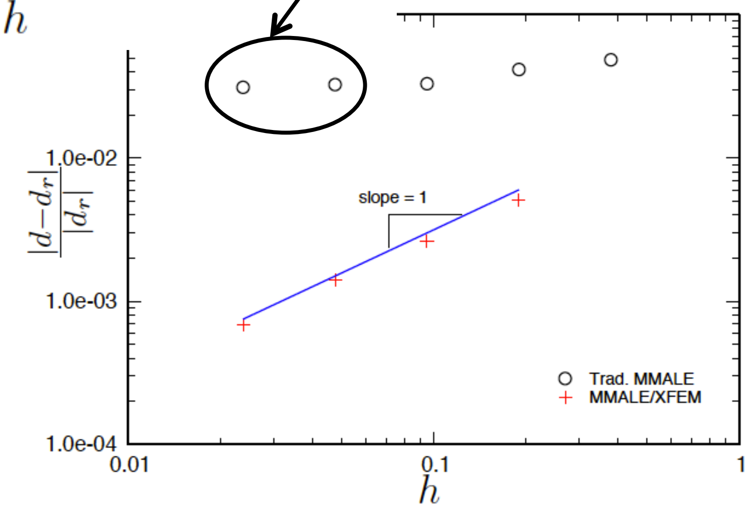


… as well as comparable rates of convergence …



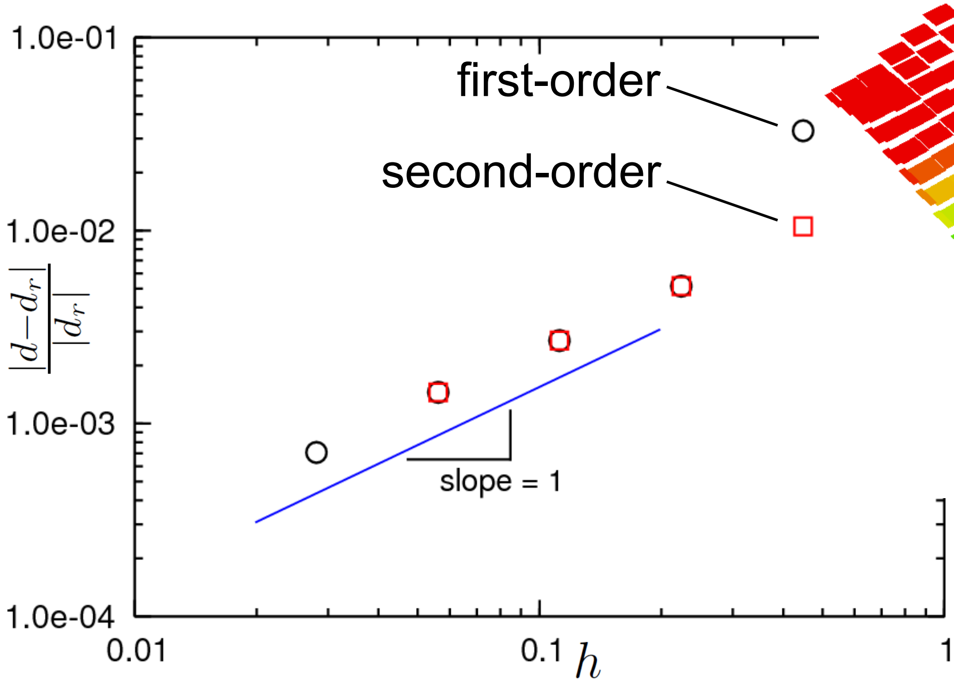
Standard Eulerian converges to a different solution.

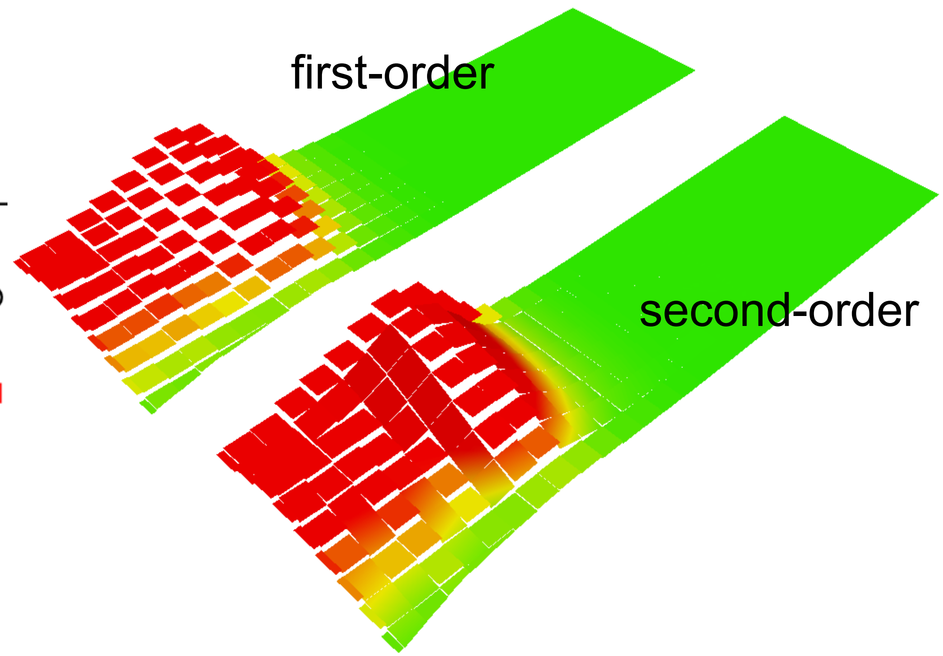




For this problem remap order has little effect on accuracy/rate-of-convergence …

Convergence in height for first- and second-order remap.

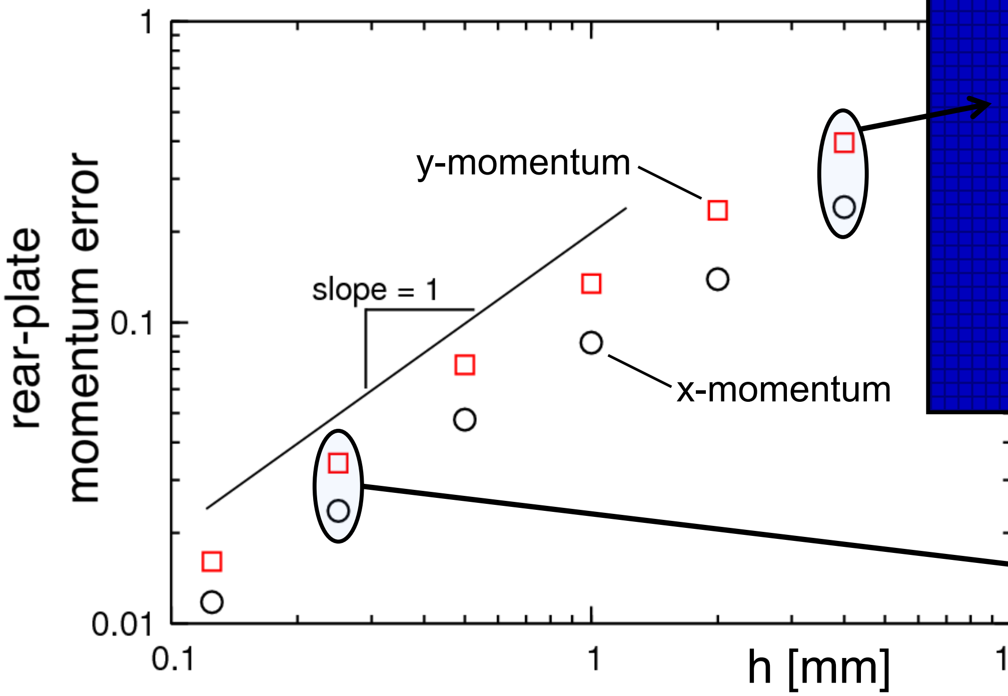


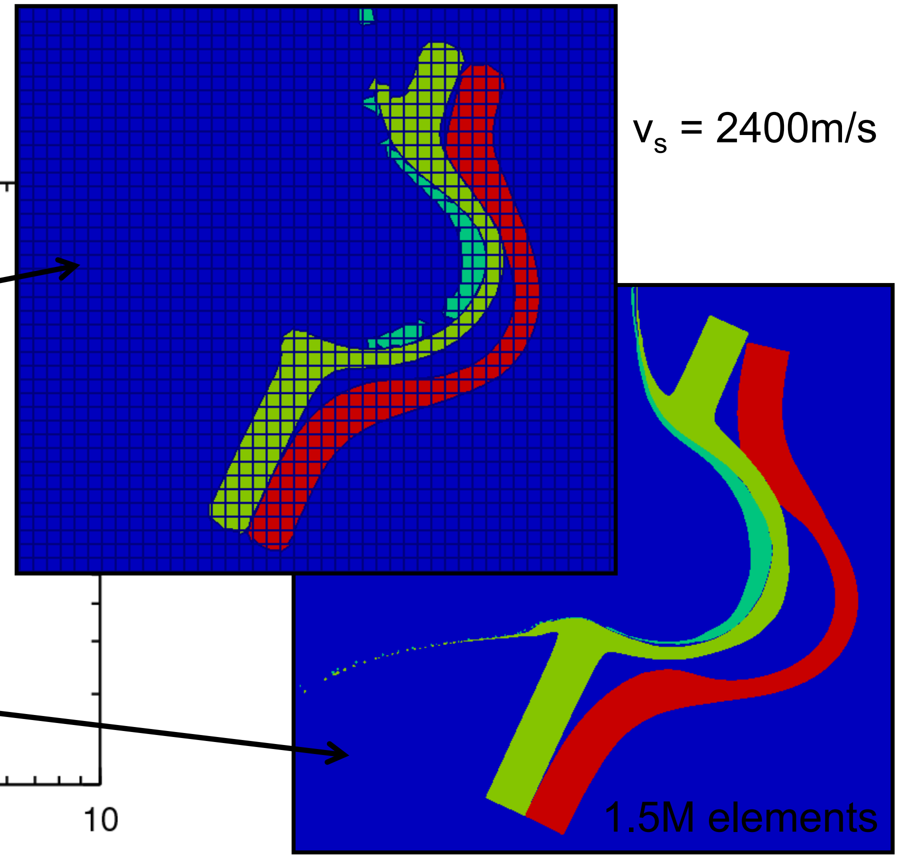


Density reconstructions at early time.

More complicated problems demonstrate the utility/advantages of the approach …

Whipple Shield used in satellite protection.





High-velocity impact difficult for Lagrangian and unrealistic for Eulerian are possible with X-FEM.

Conclusions:

- Developing capability to more accurately treat multi-material cells in an “operator-split” ALE context.
- Capability builds on existing ALE infrastructure.
- Uses X-FEM ideas to provide provide unique kinematics for each material in a cell.
- Uses interface reconstruction rather than level-set ideas to address conservation and complex interface intersections.
- Employs higher-order, conservative remapping algorithms. Advantages are unclear at this point.
- Demonstrates good convergence/accuracy for problems investigated here.

(Incomplete) References:

AHD11: Annavarapu et al., IJNME, submitted.

BMW09: Bechet et al., IJNME 78, 931.

CKT91: Carpenter et al., IJNME 32, 103.

D83: Dukowicz, JCP 54, 411.

DK87: Dukowicz and Kodis, SIAM J. Sci Stat Comput 8, 305.

DLZRM10: Dubois et al., Comp Mech 46, 329.

DMRV08: Dolbow et al., CMAME 197, 439.

HAD11: Hautefeuille et al., IJNME, in revision.

HH04: Hansbro and Hansbro, CMAME 193, 3523.

MRMCB08: Menouillard et al., IJNME 74, 447.

SAB06: Song et al., IJNME 67, 868.

VB06: Vitali and Benson, IJNME 67, 1420.

VMR10: Voth et al., USNCCM10, Columbus.